Grade Table (for checker use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Team Members:

- ....
- ....
- ....

Write your team name on top of each page.

If you have any queries, contact the invigilator. However, no questions on the validity/ correctness of a question will be entertained.

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1. (10 points) Three gods A, B, and C are called, in no particular order, $True$, $False$, and *Random*. $True$ always speaks truly, $False$ always speaks falsely, but whether *Random* speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for *yes* and *no* are *da* and ja, in some order. You do not know which word means which. What questions will you ask ?

2. (10 points) The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

3. (10 points) Let $a_1, a_2, ..., a_n$ be arbitrary real numbers. Show that the following will always hold

$$\frac{a_1}{1+a_1^2} + \frac{a_2}{1+a_1^2+a_2^2} + ... + \frac{a_n}{1+a_1^2+a_2^2+...+a_n^2} < \sqrt{n}$$

4. (10 points) We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

5. (10 points) We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task is possible when $r = 73$.

   (c) Can the task be done when $r = 97$?

6. (10 points) Let k be a positive integer and m be an odd number. Prove that there exists a natural number n such that $n^n - m$ is divisible by 2^k.

7. (10 points) A point D is chosen on side AC of triangle ABC with $\angle C < \angle A < 90^o$ in such a way that $BD = BA$. The incircle of triangle ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triange BCD. Prove that the line KL bicects line segment AJ.

8. (10 points) Let $n > 1$ be a given positive integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd.